Cosmology Short Course

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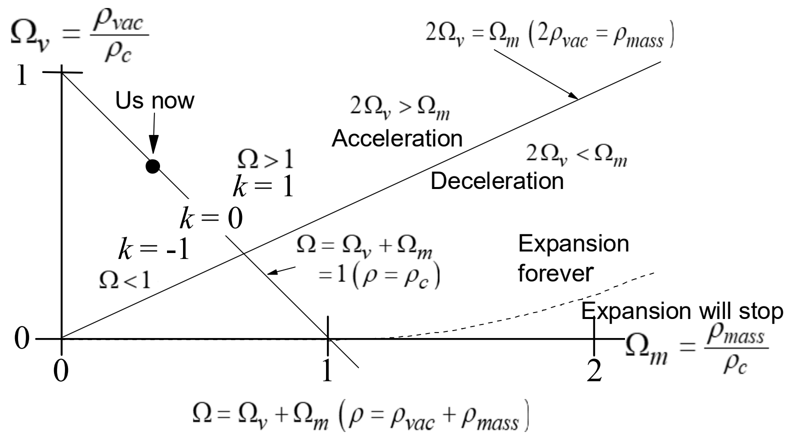
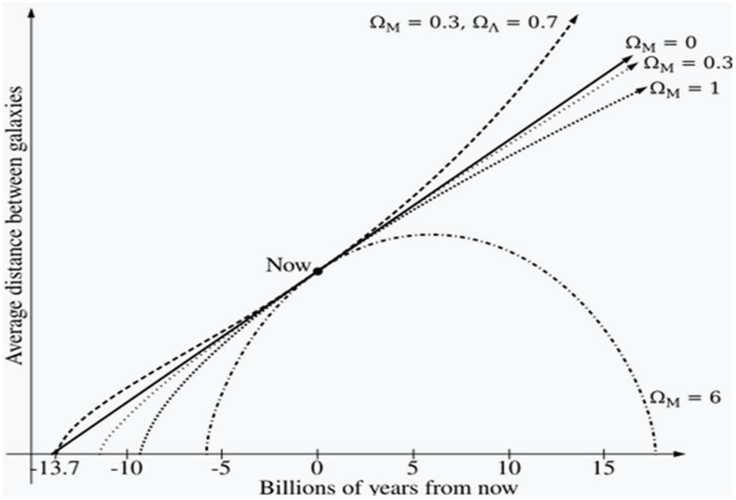
Wholeness Chart 1. Summary of Key Relations in Cosmology

Central Point	Derivation/Remark
Conservation of Energy: $d(\rho V) = -p dV$ Energy Density: $\dot{\rho} = -3(\rho + p)\dot{a}/a = -3(\rho + p)H$ $\rho(a)$ with $p = w\rho$: $\rho_w \propto 1/a^{3(1+w)}$ Rad $\rho_{w=1/3} \propto 1/a^4$ Matter $\rho_{w=0} \propto 1/a^3$ Vac $\rho_{w=-1} \propto$ const	3 ways to energy density relation: 1) Friedman/RW metric in $T^{\mu\nu}{}_{;\nu} = 0$ for $\mu = 0$. 2). $T^{\hat{0}\hat{0}}$ & $T^{\hat{x}\hat{x}}$ field eqs combined. 3) Divide top left eq by dt, use $V = ka^3$. $\rho(a)$: ρ_w into top eq, w/ $V = Ka^3$ & $p = w\rho$ $c^2\rho \to \rho$ in all this ($c = 1$)
Basic Eqs: Friedmann Universe (homogeneous, isotropic) **(A)** $\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - k\left(\frac{a_0}{a}\right)^2$ (1st) Friedmann eq. **(B)** $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$ Dynamic eq. (2nd Friedmann eq.)	Can rep $\Lambda = 8\pi\rho_{vac}$ & $\Lambda = 0$, with $\rho = \rho_{vac} + \rho_{other}$ Friedmann/RW metric in Einstein eq for T^{00} $k = 0$ (3D flat), 1 (pos curv), −1 (neg curv) Fried/RW metric in Einstein eq for T^{xx} plus (A), or energy density eq plus (A)
Radiation: $\lambda \propto a$ $\bar{E}_\gamma = 2.7k_B T \propto 1/\lambda \propto 1/a$ $n_\gamma \propto 1/a^3$ $\rho_\gamma = n_\gamma \bar{E}_\gamma = 2.7k_B n_\gamma T \propto 1/a^4$ also $p = \rho_\gamma/3$	n_γ = photon number density k_B = Boltzmann constant
***k* = 0 Time Dependence:** Radiation only: $a \propto t^{1/2}$ Matter only: $a \propto t^{2/3}$ Vacuum only: $a \propto e^{Ht}$ General: $a \propto t^{2/3(1+w)}$ except vacuum (for $p_{vac} = -\rho_{vac}$ = const)	(A) $\to$ $\dot{a} = \sqrt{\frac{8\pi G\rho}{3}}a = Ha$, use the various values of $\rho(a)$ in top left box, and solve for a.
Critical Density: $\rho_c = \frac{3H^2}{8\pi G}$ $\Omega = \rho/\rho_c = \Omega_\gamma + \Omega_m + \Omega_v$ $\Omega_i = \rho_i/\rho$ Rad (early univ) $\rho \propto 1/a^4$, $1 - \frac{1}{\Omega} \propto k a^2$ and Ω changes fast as $a \uparrow$ For inflation ρ = const, $1 - \frac{1}{\Omega} \propto \frac{k}{a^2}$ $\Omega \to 1$ ($\rho \to \rho_c$) fast with $a \uparrow$	From (A), $\Omega = 1$, if $k = 0$. (A) w/ $k = 0, 1, -1$ and $\Lambda = 0$, rearranged gives this relation in $1/\Omega$. And a grows exponentially with time for constant ρ. Approaches flat as $a \uparrow$.
Acceleration: $\ddot{a} = -\frac{4\pi G}{3}(\rho_{mass} - 2\rho_{vac})a$ $\ddot{a} > 0$ if $2\rho_{vac} > \rho_{mass}$ 	(B) for mass and vacuum, assuming vac like cosmolog constant (dark energy) with $w = -1$. More generally, in (B), if $p < -\rho/3$ (i.e., $w < -1/3$), where p and ρ are total values, get acceleration. Today: $\Omega_v = .68 \to \rho_{vac} = .68\rho_c$ ($p_{vac} = -\rho_{vac}$) $\Omega_m = .32 \to \rho_{mass} = .32\rho_c$ ($p_{mass} = 0$) $\rho = \rho_{vac} + \rho_{mass} = .68\rho_c + .32\rho_c = \rho_c$ $p = p_{vac} + p_{mass} = -.68\rho_c + 0 = -.68\rho_c$ $\rho + 3p = \rho_c - 3(.68)\rho_c = -2.04\rho_c < 0$ From (B), acceleration is positive.

Total Density	Ω	Curvature	*k*	Expansion	Accel/Decel?	Acceleration?
$\rho < \rho_c$	$\Omega < 1$	negative	$k = -1$	Never stop	Either	If $2\rho_{vac} > \rho_{mass}$
$\rho = \rho_c$	$\Omega = 1$	zero	$k = 0$	Never stop	Either	If $2\rho_{vac} > \rho_{mass}$
$\rho > \rho_c$	$\Omega > 1$	positive	$k = +1$	May stop if $\rho_{mass} > \rho_c$	Either	If $2\rho_{vac} > \rho_{mass}$



Plots of $a(t)$ vs t for various values of density.

Λ subscript refers to cosmological constant (equivalently a vacuum mass-energy density, also known as dark energy density).

Hubble Plots: <u>Def</u>: d_L = luminosity dist = distance to source if 3D flat, static universe for the luminosity we see for standard candle.

Key relation: $\frac{\lambda_0}{\lambda} = \frac{a_0}{a} = 1 + z$

Physical distance = $l_0 = \int_{a(t)}^{a_0} c \frac{a_0}{a} \frac{1}{\dot{a}} da$ expanding, curved or flat

Want to find d_L from l_0

↓ Corrections for expansion & curvature using dimensionless quantities ↓

<u>Consider Flat 3D Space (k = 0) Expanding Universe</u>

1) Luminosity decreases with $1/l_0^2$
2) " " due to photon wave stretch by $\lambda/\lambda_0 = 1/(1+z)$
3) " " due to fewer photons arriving per sec $\lambda/\lambda_0 = 1/(1+z)$

The three effects make luminosity proportional to $\frac{1}{l_0^2 (1+z)^2}$

Luminosity distance d_L = square root of denominator above

$$d_L = (1+z) l_0$$

<u>Consider Curved 3D Space (k = 1 or − 1) Expanding Universe</u>

Only 1) above is changed. For r_0 = curvature radius,

pos curv: $l_{0+1} = r_0 \sin \frac{l_0}{r_0}$ neg curv: $l_{0-1} = r_0 \sinh \frac{l_0}{r_0}$

pos curv: $d_L = (1+z) l_{0+1}$ neg curv: $d_L = (1+z) l_{0-1}$

Original Hubble plots had velocity vs distance. Now, z vs d_L (redshift z increases with velocity, d_L is a measure of distance).

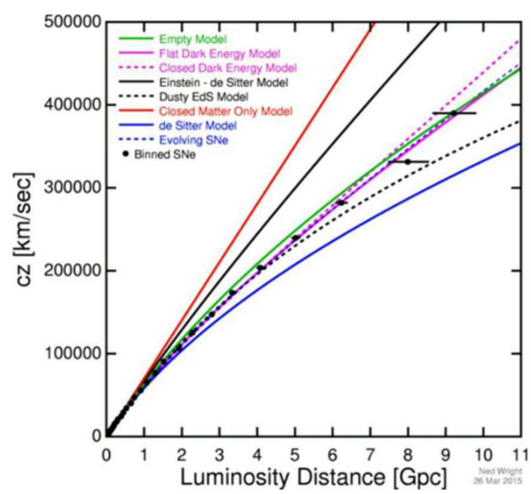
Physical distance l_0 = the actual distance at this moment that would be measured with meter sticks between us and the object. (Universe flat or curved, static or expanding universe.)

$l_0 = d_L$ only in flat, static universe.

To furthest visible region if our 3D flat universe were static l_0 = 13.7 billion light-yrs

To furthest visible region for our actual 3D flat expanding universe l_0 = 42 billion light-yrs

<u>Value:</u> z vs d_L curve depends on Ω_{m_0} and Ω_{v_0} so data helps determine them. Since universe is expanding, d_L is not a meaningful



number in terms of distance, but easy to measure.

1 Derivation of the Einstein Field Equations

See https://profoundphysics.com/derivation-of-einstein-field-equations/.

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{1}$$

2 Our Goal

We want to describe the expansion of the universe in terms of its physical size as a function of time. "Physical size" is a length parameter, but a little hard to pin down. We can't take it as the actual size of the universe, as we don't know what that size, even if it is not infinite, is. If the universe were spherically shaped (the same positive curvature everywhere in 3D space), our length parameter could be taken as the radius of the sphere. But, for a flat universe, such a radius is infinite, and thus, the concept breaks down there.

But, we can, and do, work in terms of a base length, such as the distance one would measure with meter sticks between two galactic clusters, and investigate how that changes over time, as the universe expands.

If we imagine generalized coordinate gridlines spread out over the universe at a given time, such that each galactic cluster has a set of coordinates on that grid. As time goes on and the universe expands, the grid grows in size, and the clusters keep the same coordinate numbers on the grid. But, the physical distance between the clusters increases. If we represent the coordinate number distance between the clusters by r, and the physical distance by s, we find one from the other, differentially at first, by

$$ds = a(t)\,dr \tag{2}$$

For differential distances, the factor $a(t)$ times the coordinate difference gives us the actual physical (differential) distance at a given time t. By integrating (2), we can get global distances between any two clusters in the universe.

<u>The goal</u>: Find a as a function of time.

Note that it can be more convenient to define the coordinate r as unitless, so then $a(t)$ has units of length. If we take our initial length as $a_0 = a(t_0) =$ unit length, we would have $a(t)$ itself representing the physical length to which a unit length has grown from t_0 to t. However, in general, in our equations, we retain the symbol a_0 (as not necessarily equal to 1 distance unit) and consider $a(t)/a_0$ as the ratio of length at $t > 0$ to length at $t = 0$.

3 Robertson-Walker/Friedmann Metric

Relation (2) helps for finding the distance between two places in the universe, where one is at the center of the coordinate grid being employed. It is essentially only one -dimensional. More generally, for three spatial and one time dimension, we need a relation that gives us the spacetime 4D "distance" s (or ds infinitesimally) between any two events. Robertson and Walker, and independently, Friedmann deduced this more general relationship for 3+1 dimensions.

They had three basic assumptions.

1) On the very largest scales, the universe is spatially homogeneous. At a given moment, it has same mass-energy density, pressure, temperature, entropy, and scale factor a everywhere.
2) It is isotropic. On very large scales, the universe is the same no matter what direction one looks.
3) Time t is considered measured in the following hypothetical way. At the moment of the big bang, all material points in the universe had standard clocks anchored to them. As the universe grew, these clocks all ticked along. At a later time, when all the clocks have the same reading, this is considered the same "moment" for the universe. This is the clock synchronization method used by R-W and Friedmann.

Thus, the line element ("distance" along an infinitesimal line in 4D) is, where the parameter S_k will be discussed below,

$$ds^2 = -c^2 dt^2 + a^2(t)\left(dr^2 + S_k^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right) \tag{3}$$

The metric in such a space is then

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & & & \\ & a^2(t) & & \\ & & a^2(t)S_k^2 & \\ & & & a^2(t)S_k^2 \sin^2\theta \end{bmatrix} \tag{4}$$

Note that (3) reduces to (2) along a radial line at the same point in time, i.e., when $dt = d\theta = d\phi = 0$. Note for readers of my "Metrics in Homogeneous Isotropic Curved Surfaces" article, r and R in (8) there are physically measured values. Here, r is a generalized coordinate value, obtained by taking a curved surface where the physical radius in the higher dimension R is taken as 1, and r herein in (5) is equal to r/R there with $R = 1$. We can do that here because our value of r is a generalized coordinate, so can be taken as anything we find convenient, and this is generally convenient in cosmology.

$$S_k^2 = \sin^2 r \quad k = 1 \,(\text{positive curvature}) \qquad S_k^2 = r^2 \quad k = 0 \,(\text{flat}) \qquad S_k^2 = \sinh^2 r \quad k = -1 \,(\text{negative curvature}) \quad (5)$$

4 Finding Local Energy Conservation Relation via Stress-energy Tensor

4.1 From Stress-Energy Tensor Zero Divergence

From Einstein field equation $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ and knowledge from differential geometry that the Einstein tensor $G^{\mu\nu}$ has zero divergence, i.e., $G^{\mu\nu}{}_{;\nu} = 0$, then the stress-energy tensor must also have zero divergence. For $\mu = 0$, $T^{0\nu}{}_{;\nu} = 0$ gives us (local) conservation of energy. For $\mu = i = 1,2,3$, $T^{i\nu}{}_{;\nu} = 0$ gives us conservation of momentum. We employ the $\mu = 0$ case below.

$\boxed{\text{Finding } \dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a} \text{ in Friedman metric}}$ 1, 9/16/83

$T^{\mu\nu}{}_{;\nu} = 0$ For $\mu = 0$ is conserv of en law

$T^{\mu\nu} = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$ Perf fluid (visc = 0) These are actually not phys comps as expressed. (i.e. $p^2_2 = \frac{p}{g_{22}}$, $p^1_1 = \frac{p}{a^2}$)

(1) $T^{\beta\alpha}{}_{;\gamma} = T^{\beta\alpha}{}_{,\gamma} + \Gamma^{\beta}{}_{\mu\gamma} T^{\mu\alpha} + \Gamma^{\alpha}{}_{\mu\gamma} T^{\beta\mu}$

(2) $T^{0\alpha}{}_{;\alpha} = T^{0\alpha}{}_{,\alpha} + \Gamma^{0}{}_{\mu\alpha} T^{\mu\alpha} + \Gamma^{\alpha}{}_{\mu\alpha} T^{0\mu}$

(2a) $T^{0\alpha}{}_{;\alpha} = T^{00}{}_{,0} + 0 + 0 + 0 + \Gamma^{0}{}_{00} T^{00} + \Gamma^{0}{}_{11} T^{11} + \Gamma^{0}{}_{22} T^{22} + \Gamma^{0}{}_{33} T^{33}$
$+ \Gamma^{0}{}_{00} T^{00} + \Gamma^{1}{}_{01} T^{00} + \Gamma^{2}{}_{02} T^{00} + \Gamma^{3}{}_{03} T^{00}$

$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \Sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$ $\Sigma = \chi$ for $k = 0$ $\quad$ Σ here $= S_k$ in (5); χ here $= r$ in (5)

$g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & a^2 & & \\ & & a^2\Sigma^2 & \\ & & & a^2\Sigma^2\sin^2\theta \end{bmatrix}$ $\quad g^{\mu\nu} = \begin{bmatrix} -1 & & & \\ & \frac{1}{a^2} & & \\ & & \frac{1}{a^2\Sigma^2} & \\ & & & \frac{1}{a^2\Sigma^2\sin^2\theta} \end{bmatrix}$

$\Gamma_{\delta\beta\gamma} = \frac{1}{2}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right)$ $\quad \Gamma^{\mu}{}_{\beta\gamma} = g^{\mu\delta}\Gamma_{\delta\beta\gamma}$

$\Gamma_{011} = \frac{1}{2}(\cancel{g_{01,1}} + \cancel{g_{01,1}} - g_{11,0}) = \frac{1}{2}(-2a\dot{a}) = -a\dot{a}$

$\Gamma_{111} = \frac{1}{2}(g_{11,1} + \cancel{g_{11,1} - g_{11,1}}) = 0$

$\Gamma_{211} = \frac{1}{2}(\cancel{g_{21,1}} + \cancel{g_{21,1}} - \cancel{g_{11,2}}) = 0$

$\Gamma_{311} = 0$

$\Gamma^{0}{}_{11} = g^{00}\Gamma_{011} = \dot{a}a$ so $\Gamma^{0}{}_{11}T^{11} = a\dot{a}\,p^1_1$ $\quad$ For phys momentum comps $\boxed{\Gamma^{\hat{0}}{}_{\hat{1}\hat{1}}T^{\hat{1}\hat{1}} = a\dot{a}\frac{p}{a^2} = \frac{\dot{a}}{a}p}$

$\boxed{T^{00}{}_{,0} = \frac{\partial\rho}{\partial t}}$ $\quad \Gamma_{000} = 0$ so $\boxed{\Gamma^{0}{}_{00}T^{00} = 0}$

Phys comps in θ & ϕ direcs will be same as due to isotropy so $\boxed{\Gamma^{\hat{0}}{}_{\hat{2}\hat{2}}T^{\hat{2}\hat{2}} = \Gamma^{\hat{0}}{}_{\hat{3}\hat{3}}T^{\hat{3}\hat{3}} = \frac{\dot{a}}{a}p}$

2.

$$\Gamma_{000} = 0$$

$$\Gamma_{101} = \tfrac{1}{2}\left(\cancel{g_{10,1}} + g_{11,0} - \cancel{g_{01,1}}\right) = a\dot{a}$$

$$\Gamma_{201} = \tfrac{1}{2}\left(\cancel{g_{20,1}} + \cancel{g_{21,0}} - \cancel{g_{01,2}}\right) = 0$$

$$\Gamma_{301} = 0$$

$$\Gamma_{100} = \tfrac{1}{2}\left(\cancel{g_{10,0}} + \cancel{g_{10,0}} - \cancel{g_{00,1}}\right) = 0$$

$$\Gamma_{200} = \tfrac{1}{2}\left(0 + 0 - \cancel{g_{00,2}}\right) = 0$$

$$\Gamma_{300} = 0$$

$$\Gamma^{0}_{00} = g^{00}\Gamma_{000} = 0 \qquad \boxed{\Gamma^{0}_{00}T^{00} = (0)\rho = 0}$$

$$\Gamma_{001} = \tfrac{1}{2}\left(\cancel{g_{00,1}} + \cancel{g_{01,0}} - \cancel{g_{01,0}}\right) = 0$$

$$\Gamma_{101} = \tfrac{1}{2}\left(\cancel{g_{10,1}} + g_{11,0} - \cancel{g_{01,1}}\right) = a\dot{a}$$

$$\Gamma_{201} = \tfrac{1}{2}\left(0 + 0 - 0\right) = 0$$

$$\Gamma_{301} = \tfrac{1}{2}\left(0 + 0 - 0\right) = 0$$

$$\Gamma^{1}_{01} = g^{10}\Gamma_{001} = g^{11}\Gamma_{101} = \frac{\dot{a}}{a} \qquad \boxed{\Gamma^{1}_{01}T^{00} = \frac{\dot{a}}{a}\rho}$$

This is phys comp.

By sym, phys comps in y & z direcs → $\boxed{\underbrace{\Gamma^{2}_{02}T^{00}}_{\text{phys}} = \underbrace{\Gamma^{3}_{03}T^{00}}_{\text{phys}} = \frac{\dot{a}}{a}\rho}$

subst □ vals in (2a)

$$(3) \qquad \boxed{\frac{\partial \rho}{\partial t} = -3(\rho + p)\frac{\dot{a}}{a}} \qquad \text{QED.}$$

H.W. Problem #1: Derive handwritten (3) above using the energy balance relation $d(\rho V) = -p\,dV$ where $V \propto a^3$.

H.W. Problem #2: Show that the energy balance equation holds for 1) for mass, where $p = 0$, $\rho \propto 1/a^3$; 2) for radiation, where $p = \rho/3$, $\rho \propto 1/a^3$ 3) for the vacuum, assuming $p = -\rho$, $\rho =$ constant, and 4) for any substance where the equation of state is $p = w\rho$ (w a constant), $\rho \propto a^{3(1+w)}$.

Note: Mass-energy in a space that expands like our universe (or, hypothetically, changes in other ways) is actually not conserved globally (other than in a differential volume). To see this by way of example, consider photons in our expanding universe. Their energy density decreases as volume increases. The same number of photons in a given co-moving volume (which is expanding) occupies a larger physical volume as time increases. The physical volume changes in proportion to a^3, so the energy density decreases by $1/a^3$. If that were all there were to it, total energy would remain constant.

However the photon wavelength is stretched as the universe expands, which means its frequency decreases. Since, for a single photon, $E = hf$, its energy decreases with the expansion. Hence, energy is not conserved globally, even though it is locally by the energy balance relation derived above.

5 Equation of State for Radiation and Vacuum

5.1 Radiation (Photons)

From Jordan[1]:

APPENDIX A: RADIATION PRESSURE

Consider the pressure on a small plane area A from photons of momentum $\vec{p}$ reflecting off it. Let p_x be the component of $\vec{p}$ perpendicular to A for a photon approaching A. Reflection changes p_x to $-p_x$ and leaves the other components of $\vec{p}$ unchanged. Force is the rate of change of momentum, so the force on A is $2p_x$ times the number of photons that hit A per unit time. Altogether, including photons with different momenta $\vec{p}$, the force on A is

$$\tfrac{1}{2}2\overline{p_x v_x} A n_\gamma , \tag{A1}$$

where v_x is the x component of the velocity of a photon with momentum $\vec{p}$ and n_γ is the number of photons per unit volume. The average is over all $\vec{p}$, but only photons with positive p_x exert force on A. They are just half of the photons present; that explains the factor $\frac{1}{2}$. The velocity of a photon is c in the direction of its momentum. We assume there is a uniform distribution of momentum directions. Then the pressure is

$$\begin{aligned} p_\gamma &= \overline{p_x v_x} n_\gamma = \overline{p_x (p_x/|\vec{p}|)} c n_\gamma \\ &= c\frac{1}{3}\frac{\overline{p_x^2+p_y^2+p_z^2}}{|\vec{p}|} n_\gamma = \frac{1}{3}c\overline{|\vec{p}|} n_\gamma = \frac{1}{3}\rho_\gamma , \end{aligned} \tag{A2}$$

because the average does not depend on the direction of the momentum and the energy of a photon is

$$h\nu = c\frac{h}{\lambda} = c|\vec{p}| . \tag{A3}$$

Equation of state $p = w\rho$

For photons $w = 1/3$

5.2 The Vacuum

Assumptions: The vacuum is Lorentz invariant, and behaves like a perfect fluid with stress-energy tensor (6). Note that $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transformation; Λ is the cosmological constant.

$$T_{\mu\nu} = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} \qquad \Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & & \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad T'_{\mu\nu} = \Lambda^{\beta}{}_{\nu}\Lambda^{\alpha}{}_{\nu} T_{\alpha\beta} \tag{6}$$

$$T'_{00} = \frac{\rho + v^2 p}{1-v^2} \qquad T'_{11} = \frac{\rho + v^2 p}{1-v^2} \tag{7}$$

Only way $T'_{\mu\nu} = T_{\mu\nu}$ is if $p = -\rho$. Thus, thus, for the equation of state for the vacuum,

$$p = w\rho \qquad \to \qquad p_{vac} = -\rho_{vac} \qquad w = -1 \text{ for vacuum} \tag{8}$$

And, for the vacuum.

$$T_{\substack{\mu\nu \\ vac}} = \begin{bmatrix} \rho_{vac} & & & \\ & -\rho_{vac} & & \\ & & -\rho_{vac} & \\ & & & -\rho_{vac} \end{bmatrix} = -\rho_{vac}\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = -\rho_{vac}\eta_{\mu\nu} = -\rho_{vac} g_{\mu\nu} \left(\text{for local co-moving observer}\right) \tag{9}$$

Note: With (9), we can re-express Einstein's field equation (1) as

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} - \frac{\Lambda\rho_{vac}}{8\pi G} 8\pi G \frac{g^{\mu\nu}}{\rho_{vac}} = 8\pi G T^{\mu\nu} + \frac{\Lambda}{8\pi G\rho_{vac}} 8\pi G T^{\mu\nu}_{vac} . \tag{10}$$

For a cosmological constant

$$\Lambda = 8\pi G\rho_{vac} \qquad \to \qquad \rho_{vac} = \frac{\Lambda}{8\pi G} \text{ in } T^{\mu\nu}_{vac} , \tag{11}$$

(10) becomes

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} + 8\pi G T^{\mu\nu}_{vac} = 8\pi G T^{\mu\nu}_{effective\,total} . \tag{12}$$

In other words, whatever cosmological constant Λ the universe might have, we can model it as vacuum energy and pressure by using the RHS of (11) for ρ_{vac} and using that in Einsteins' field equation as the vacuum contribution $T^{\mu\nu}_{vac}$ to the total stress energy tensor. We will sometimes, but not always, do this in what follows and when doing so, not write the subscript "*effective total*" on that tensor. In some cases, this will make things easier. In others, it will be easier to think in terms of a Λ term. But keep in mind that we can switch back and forth, and this is common in the literature.

6 The Hubble "Constant"

Cosmological objects recede from us, and the further away they are, the faster the rate of recession (speed). As a balloon with ants on it expands, each ant sees other ants moving away from it, and the further away on the surface the ants are from a given observer ant, the more rapidly they are moving relative to the observer. Similar for our universe.

This increase in speed with distance is quantified in the Hubble constant H, where a parsec equals 3.26 light-years in distance and a megaparsec (Mparsec) is a million times that,

$$H = \frac{\text{km/sec}}{\text{Mparsec}} \qquad \left(\text{between 67 and 74 numerically from experiment as of 2025}\right) \tag{13}$$

As the universe expands, the expansion rate changes. It was once thought (prior to 1998) that the expansion would slow (decelerate) with time, but now we know that it is, in fact, speeding up (accelerating). So, H actually changes over time. It is not constant, despite, for historical reasons, being called the Hubble "constant". However, at a given point in time, it is considered constant over space. It is the same for all observers anywhere in the universe at the same moment.

As we look out through our telescopes, however, since light from objects takes time to reach us, we see other objects as they were in the past. The further out we look, the further back what we are viewing is in the past, and so the Hubble constant looks different for those objects, since it changes in time. If, however, we could look at those objects via some imaginary surrogate means that propagated at infinite speed, we would see them as they are now, and the Hubble constant for them would be the same as it is for us.

We could, of course, write (13) with any units we choose, such as m/sec divided by m. It is the same Hubble constant, just expressed in different systems of measurement. So, consider the physical distance l between some cosmological object and us as expressed in the co-moving coordinate r and $a(t)$ of the Robertson-Walker metric. Note that r to a given object is a fixed value as the universe expands, while the physical distance to it increases. At a fixed point in time,

$$H = \frac{dl/dt}{l} = \frac{d(ar)/dt}{ar} = \frac{r\,da/dt}{ar} = \frac{r\dot{a}}{ar} \quad \rightarrow \quad \boxed{H = \frac{\dot{a}}{a}}\,. \tag{14}$$

(14) is a relation used repeatedly in cosmology. Commit it to memory.

7 The Friedman Equations

7.1 1st Friedman Equation (often just "the Friedman equation")

7.1.1 The Easy Way to Almost Derive

Jordan[1], in his first two pages, derives the first Friedmann equation using Newtonian mechanics, and actually gets the same result as in GR, though one would have to do the full derivation (next sub-section) in GR to know that Jordan's result is correct for GR. (Note his R is what we use a for.)

7.1.2 The Hard Way to Derive

Use the Friedmann/Robertson-Walker metric in the Einstein field equation for T^{00}. Need to find Christoffel symbols, evaluate derivatives, etc. This is lengthy, and we won't do it here.

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - k\left(\frac{a_0}{a}\right)^2} \quad \text{from } G_{00} = 8\pi G T_{00} \quad 1^{\text{st}} \text{ Friedman equation (or the Friedmann equation)} \tag{15}$$

7.2 2nd Friedman Equation (often "the dynamic equation")

7.2.1 The Easy Way to Derive

Take the time derivative of the first Friedman equation and use the (local) energy conservation equation (second row in Wholeness Chart 1, pg. 1).

$$\frac{d}{dt}\left(\frac{\dot a}{a}\right)^2=\frac{d}{dt}\left(\frac{8\pi G\rho}{3}+\frac{\Lambda}{3}-k\left(\frac{a_0}{a}\right)^2\right)\;\rightarrow\;2\left(\frac{\dot a}{a}\right)\left(\frac{1}{a}\frac{d}{dt}\dot a+\dot a\frac{d}{dt}\frac{1}{a}\right)=\frac{8\pi G\dot\rho}{3}-2k\left(\frac{a_0}{a}\right)\frac{d}{dt}\left(\frac{a_0}{a}\right)$$

$$\left(\frac{\dot a}{a}\right)\left(\frac{\ddot a}{a}+\dot a\left(\frac{-1}{a^2}\right)\dot a\right)=\frac{4\pi G\dot\rho}{3}-k\left(\frac{a_0}{a}\right)a_0\left(\frac{-1}{a^2}\right)\dot a\;\rightarrow\;H\left(\frac{\ddot a}{a}-\left(\frac{\dot a}{a}\right)^2\right)=\frac{4\pi G\dot\rho}{3}+k\left(\frac{a_0}{a}\right)^2H \qquad (16)$$

$$\frac{\ddot a}{a}-H^2=\frac{4\pi G\dot\rho}{3H}+k\left(\frac{a_0}{a}\right)^2\;\rightarrow\;\frac{\ddot a}{a}=\frac{4\pi G\dot\rho}{3H}+H^2+k\left(\frac{a_0}{a}\right)^2$$

Now, use the energy density balance equation (2nd equation in first chart) for the time derivative of ρ.

$$\frac{\ddot a}{a}=\frac{4\pi G}{3H}\big(-3(\rho+p)H\big)+H^2+k\left(\frac{a_0}{a}\right)^2 \qquad (17)$$

Then, use (15) for H^2.

$$\begin{aligned}\frac{\ddot a}{a}&=\frac{4\pi G}{3}\big(-3(\rho+p)\big)+\frac{8\pi G\rho}{3}+\frac{\Lambda}{3}-k\left(\frac{a_0}{a}\right)^2+k\left(\frac{a_0}{a}\right)^2\\&=-3\frac{4\pi G}{3}\rho-3\frac{4\pi G}{3}p+\frac{4\pi G(2\rho)}{3}+\frac{\Lambda}{3}\end{aligned} \qquad (18)$$

$$\boxed{\frac{\ddot a}{a}=-\frac{4\pi G}{3}(\rho+3p)+\frac{\Lambda}{3}}.\qquad \text{2nd Friedmann equation (or dynamic equation)}$$

7.2.2 The Harder Way to Derive

Use the Friedmann/Robertson-Walker metric in the Einstein field equation for $G_{11}=8\pi GT_{11}$ (which is the same for T_{22} and T_{33} due to isotropy) and use either 1) the 1st Friedmann equation (15) or 2) the (local) energy density balance equation (second row in Wholeness Chart 1, pg. 1).

8 Time Dependence for 3D Flat (k = 0) Universe As Ours Seems to Be

Consider the first Friedman equation (15) for k = 0, and Λ modeled as energy density and pressure in the stress-energy equation. We examine each type of material (radiation, mass, vacuum) alone without the others. This can have practical value, as the early universe was dominated by radiation (so matter and vacuum energy densities are insignificant), and the far future universe will be dominated by the vacuum (so matter and radiation can be ignored). Thus, (15) becomes

$$\dot a=\sqrt{\frac{8\pi G\rho}{3}}a\quad\rightarrow\quad\frac{\dot a}{a}=H=\sqrt{\frac{8\pi G\rho}{3}}\qquad \text{Flat, }\Lambda=0\text{ or modeled as included in }T^{\mu\nu} \qquad (19)$$

8.1 Radiation Alone

For photons, we know that energy density is diluted by expansion of volume, i.e., by a factor of $1/a^3$, and further diluted by a stretching of the wavelength by a factor of $1/a$. So, photon energy density varies with $1/a^4$. Thus, (19) gives us, where C is a constant,

$$\dot a\propto\sqrt{\frac{8\pi G}{3}\frac{1}{a^4}}a\qquad \dot a=C\sqrt{\frac{8\pi G}{3}}\frac{1}{a} \qquad (20)$$

The solution, checked via substitution, is, where K is a constant,

$$a=Kt^{1/2}\;\rightarrow\;\dot a=\tfrac{1}{2}K\frac{1}{t^{1/2}}=C\sqrt{\frac{8\pi G}{3}}\frac{1}{t^{1/2}}\qquad K=2C\sqrt{\frac{8\pi G}{3}}\qquad a=C\sqrt{\frac{32\pi G}{3}}t^{1/2}\qquad a\propto t^{1/2} \qquad (21)$$

8.2 Matter Alone

H.W. Problem #3. Show that, for matter, $a\ \alpha\ t^{2/3}$

8.3 Vacuum Alone

From (19), where the vacuum energy density is constant, and there is no mass-energy or Λ in the universe,

$$\dot{a}=\sqrt{\frac{8\pi G\rho_{vac}}{3}}a=Ha \qquad \rho_{vac},H \text{ constants} \tag{22}$$

So,

$$a=\underset{\sim}{K}e^{Ht} \qquad \underset{\sim}{K} \text{ a constant .} \tag{23}$$

If vacuum energy density is not zero, we get exponential expansion with time. If it is zero, $\dot{a}=0$ and we have a static universe.

H.W. Problem #4. Show that for a radiation dominated flat universe, even though its size increases toward infinity as $t\to\infty$, the speed of expansion goes to zero. Then, show the same thing happens, qualitatively, though at different rates, for a matter dominated flat universe. Things are different for a curved universe where $k\neq 0$, since (19) no longer holds. It has an extra term with a factor of k in it.

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9 Critical Density

9.1 Definition

In the first Friedman equation (15) for a flat universe, where $k = 0$, and we model a cosmological constant as part of the stress-energy tensor,

$$H^2=\frac{8\pi G\rho}{3} \qquad \rho=\rho_{mass}+\rho_{\gamma}+\rho_{vac}\,. \tag{24}$$

Thus, for a flat universe, we must have a mass-energy density ρ which solves (24). We call this the critical density ρ_c.

$$\rho_c=\frac{3H^2}{8\pi G}=\text{density universe must have if flat (critical density).} \tag{25}$$

9.2 Curved Space

Note if $k\neq 0$, then (15), with (25), becomes

$$H^2+k\left(\frac{a_0}{a}\right)^2=\frac{8\pi G\rho}{3} \quad\to\quad \rho=\frac{3H^2}{8\pi G}+\frac{3}{8\pi G}k\left(\frac{a_0}{a}\right)^2=\rho_c+\frac{3}{8\pi G}k\left(\frac{a_0}{a}\right)^2 \tag{26}$$
$$\text{For } k=1,\ \rho>\rho_c \qquad \text{For } k=-1,\ \rho<\rho_c$$

Result: For a positively curved universe (k=1), mass-energy density must be greater than the critical density. For a negatively curved universe, it must be less than the critical density. Conversely, the reverse is true. If density is greater than critical, 3D space is positively curved; if less, 3D space is negatively curved.

9.3 Ratio of Actual Density to Critical

We define the symbol Ω as the ratio of actual to critical mass-energy density, along with Ω_γ, Ω_m, and Ω_v, as the ratios of radiation, mass, and vacuum mass-energy densities, respectively.

$$\Omega=\frac{\rho}{\rho_c} \qquad \Omega_\gamma=\frac{\rho_\gamma}{\rho_c} \qquad \Omega_m=\frac{\rho_{mass}}{\rho_c} \qquad \Omega_v=\frac{\rho_{vac}}{\rho_c} \tag{27}$$

$$\Omega=\Omega_\gamma+\Omega_m+\Omega_v=\frac{\rho_\gamma+\rho_{mass}+\rho_{vac}}{\rho_c} \qquad =1 \text{ if } \rho_\gamma+\rho_{mass}+\rho_{vac}=\rho_c\,. \tag{28}$$

9.4 Relations with Ω

9.4.1 Summary of what Ω tells us

$$\Omega=1 \text{ or flat 3D universe } (k=0) \qquad \Omega>1 \text{ for positive curvature } (k=1) \qquad \Omega<1 \text{ for negative curvature } (k=-1) \tag{29}$$

9.4.2 Informative relation with Ω

From (26), we have

$$\rho = \rho_c + \frac{3}{8\pi G}k\left(\frac{a_0}{a}\right)^2 \quad\rightarrow\quad \frac{\rho_c}{\rho} = \frac{\rho}{\rho} - k\frac{3}{8\pi G}\left(\frac{a_0}{a}\right)^2\frac{1}{\rho} \quad\rightarrow\quad 1-\frac{1}{\Omega} = k\frac{1}{a^2\rho}\frac{3a_0^2}{8\pi G}$$
$$1-\frac{1}{\Omega} \propto k\frac{1}{a^2\rho} \tag{30}$$

If k = 0, Ω = 1. Again, we see flatness implies critical density and vice versa.

If k = 1, the RHS of (30) is positive, so we must have Ω > 1, as we saw before.

If k = −1, the RHS of (30) is negative, so we must have Ω < 1, as we saw before.

In the chart on page 1, the block titled "Critical Density", we summarize how, in the early universe, when radiation dominated and where $\rho \,\alpha\, 1/a^4$ in (30), Ω (and hence ρ) gets small rapidly (changes with a^2).

For inflation, in ways we haven't seen in this document, mass-energy density is constant, so the RHS of (30) varies with $1/a^2$. Hence, as the universe expands (exponentially in time as we saw in (23)), Ω→1. You may have heard that inflation turns the curvature of the universe, no matter what it might be, to 3D flat, and in a hurry. This is the math behind how that happens.

H.W. Problem #5. Show how the dynamics and geometry of the early universe (post inflation) must have been dominated by radiation mass-density. Then show how, subsequently, the universe, for quite some time, was dominated by mass. And how, in the far future, it will be dominated by the cosmological constant (source = dark energy), or in other words, by vacuum mass-energy density (at least it can be modeled that way).

H.W. Problem #6. Start with the 2nd Friedmann equation (18), but assume the cosmological constant part with Λ is included in the mass-energy density ρ and pressure p, so there is no Λ term. Then show, from that equation, that we get

Universe deceleration if ρ + 3p > 0. Universe acceleration if ρ + 3p < 0.

Then, show, if we have positive energy density for the vacuum (i.e., for dark energy, positive Λ) with no mass, then we have negative pressure, and the universe will accelerate.

Then show, for the present day, where mass density is 32% and dark energy density is 68%, that we have acceleration.

Then show for what values of p, relative to ρ, do we have acceleration or deceleration.

Then, note how this shows up in Fig. 1 below (and also on pg. 1 near the bottom).

9.5 Relevant Plots

9.5.1 Mass-Energy Densities

We summarize the above results in Fig. 1 and Wholeness Chart 2 below, which are repeated in the summary chart of pg. 1.

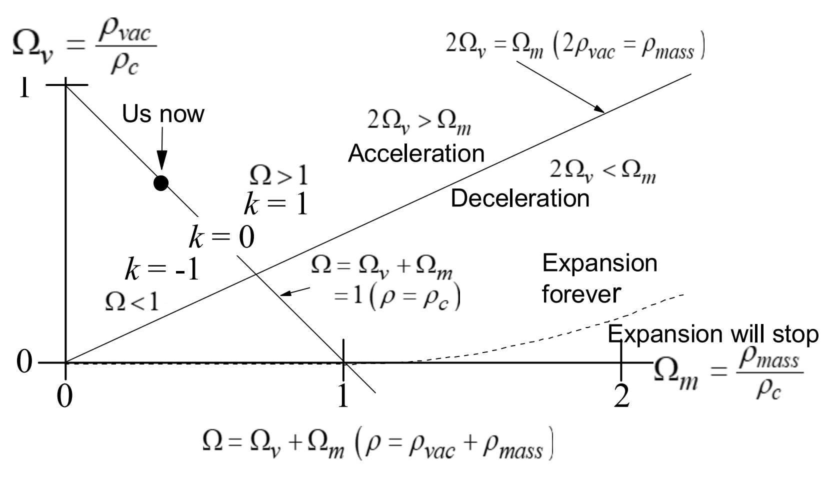


Figure 1. Universe Behavior Depending on Mass-Energy Densities of Various Constituents

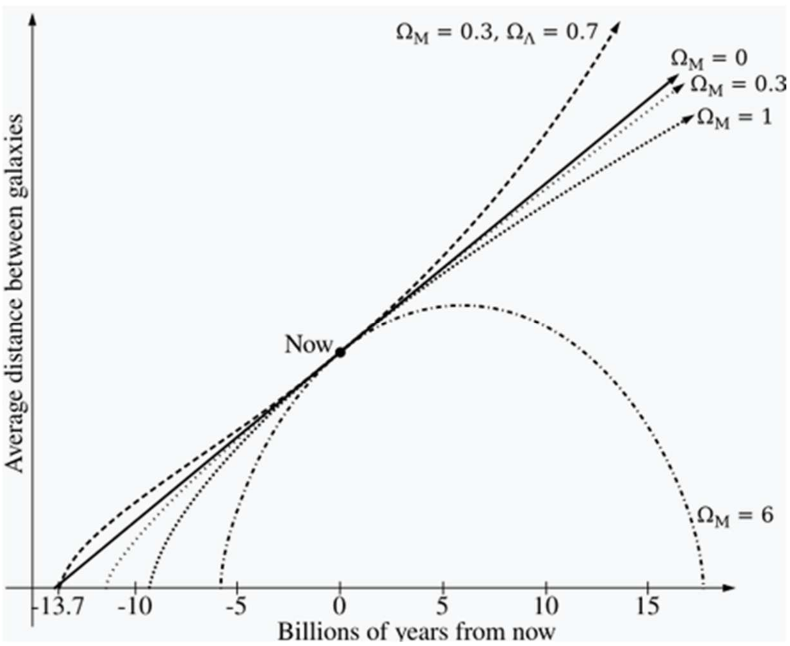
Wholeness Chart 2. Effects of Various Densities on Universe

Total Density	Ω	Curvature	k	Expansion	Accel/Decel?	Acceleration?
$\rho < \rho_c$	$\Omega < 1$	negative	$k = -1$	Never stop	Either	If $2\rho_{vac} > \rho_{mass}$
$\rho = \rho_c$	$\Omega = 1$	zero	$k = 0$	Never stop	Either	If $2\rho_{vac} > \rho_{mass}$
$\rho > \rho_c$	$\Omega > 1$	positive	$k = +1$	May stop if $\rho_{mass} > \rho_c$	Either	If $2\rho_{vac} > \rho_{mass}$

9.5.2 Expansion History for Various Mass and Vacuum Densities

Solving the Friedmann equations for $a(t)$ one gets plots like those shown in Fig. 2, for various values of mass-energy density.

From Wikipedia



The age and ultimate fate of the universe can be determined by measuring the Hubble constant today and extrapolating with the observed value of the deceleration parameter, uniquely characterized by values of density parameters (Ω_M for matter and Ω_Λ for dark energy).
A **closed universe** with $\Omega_M > 1$ and $\Omega_\Lambda = 0$ comes to an end in a Big Crunch and is considerably younger than its Hubble age.
An **open universe** with $\Omega_M \leq 1$ and $\Omega_\Lambda = 0$ expands forever and has an age that is closer to its Hubble age. For the accelerating universe with nonzero Ω_Λ that we inhabit, the age of the universe is coincidentally very close to the Hubble age.

Figure 2. Various Expansion Histories for Universes with Different Constituents

10 Hubble Plots

10.1 Background

10.1.1 Historical Plotting

In the early days of study of the Hubble constant (13), charts of it plotted speed (of recession) vs distance. However, the things that are actually measured via observation are redshift, which is related to speed, and luminosity, which is related to distance (the further away, the dimmer). So, nowadays, Hubble charts plot redshift vs luminosity distance. There are subtleties in this, which we explore in the following.

10.1.2 A Key Concept and Associated Relations

Note that redshift, symbolized by z, is a change in wavelength $\Delta\lambda$ from the original wavelength λ.

$$\boxed{z = \frac{\Delta\lambda}{\lambda}} \qquad (31)$$

In the course of expansion, the wavelength stretches with a. If a doubles, then λ doubles. One can envision this by considering the peaks of a sine wave traveling as the universe expands. Two successive peaks act as two objects ejected from a distant source would with a given separation distance between them. If the universe were static, the two objects would pass by us with the same separation between them and the same arrival time difference between them as their emission time difference. If the universe is expanding, however, the second object would take longer to reach us than the first, i.e., the separation between them would increase. Such is the case with subsequent peaks in a sine wave. Their separation increases, i.e., the wavelength is stretched.

Now, consider λ the original wavelength of light earlier in time when emitted; and λ_0, the wavelength at present when it is seen by us. Then,

$$\frac{a_0}{a}=\frac{\lambda_0}{\lambda}=\frac{\lambda+\Delta\lambda}{\lambda}=1+\frac{\Delta\lambda}{\lambda}=1+z \quad \rightarrow \quad \boxed{\frac{a_0}{a}=1+z} \tag{32}$$

Best to commit (31) and (32) to memory, as they show up often in cosmology literature.

10.2 Physical Reception Distance

10.2.1 Flat Space

Consider first a flat space, as depicted on the LHS of Fig. 3. The symbol l represents actual physical distance between two objects as time evolves from t to t_0 (the past to the present). We are located on the Earth at the right arrowhead; the object we are observing (a supernova typically), at the left arrowhead.

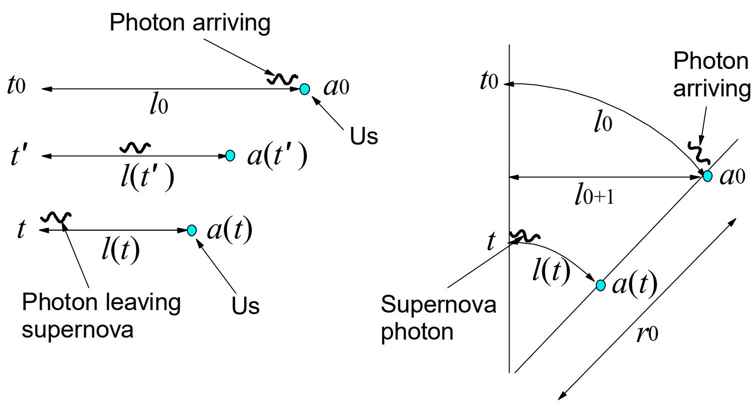


Figure 3. Change in Physical Length Between Objects with Time: Flat vs Positively Curved 3D
(Same co-moving coordinate difference between objects for all time.)

The co-moving coordinate distance r between us and the object, remains constant, while the physical distance l is a function of time. We label the earlier time without a prime, the intermediate time with a prime, and the final time, the present, with subscript zero. In general, for any time t', where $t \le t' \le t_0$,

$$l(t') = a(t')r\ . \tag{33}$$

So, we could find our final length via

$$l(t_0) = l_0 = a(t_0)r\ . \tag{34}$$

However, we would like to express this result in terms mass-energy densities, rather than a and r. To do this consider that if space were *not* expanding, where we include the symbol c for non-natural units application,

$$l_0 = \int_t^{t_0} c\,dt' = l(t') \quad t < t_0 \qquad \text{for static universe}\ . \tag{35}$$

However, since the universe is expanding, the distance traveled by light emitted from the source (LHS diagram in Fig. 3) is greater than $l(t)$. As the photon travels, the distance between the object and us increases. This increase is most when t' is just barely greater than t and least just before the photon reaches us, when t' is just barely less than t_0. At each point in time t' the stretch yet to be encountered by the photon will increase by $a_0/a(t') > 1$. That modifies (35) to

$$l_0 = \int_t^{t_0} \frac{a_0}{a(t')} c\,dt' = l(t) \quad t < t_0 \qquad \text{for expanding universe}\ . \tag{36}$$

We can re-write (36) as

$$l_0 = \int_t^{t_0} \frac{a_0}{a}\frac{c\,dt'}{da}da = \int_{a(t)}^{a_0} c\frac{a_0}{a}\frac{1}{\dot a}da\ . \tag{37}$$

By solving the 1st Friedmann equation (15) for $\dot a$ and substituting in (37), we can get, at least in principle, l_0 as a function of curvature k and the various mass-energy densities ρ_i. Numerical integration can find actual values.

10.2.2 Curved Space

Consider a positively curved space, expanding as depicted in the RHS diagram of Fig. 3, where we suppress two dimensions of spaee. Note that 3D space is positively curved with respect to a fourth spatial (not time) dimension, similar to a ball in 3D whose surface is 2D curved with respect to a third spatial dimension.

l_0 in this space is found via the same logic as we used to obtain both (34) and (37), and those relations are good for it as well, to find physical distance at present between us and the object being viewed.

10.2.3 Our Universe

If our present universe were static, for a time period of 13.7 billion years, we would find the distance to the furthest regions we can see (the furthest distance from which light was emitted at the big bang) to be l_0 = 13.7 billion light-years. For our actual expanding universe, that distance has grown much greater, and we find l_0 = 42 billion light-years.

10.3 Luminosity Distance

10.3.1 The Basic Idea

Given an intrinsic brightness of an object, measured in energy/sec emitted, the luminosity ("brightness" we would see) diminishes with distance. Luminosity is a measure of energy/sec per unit area. As light from a source such as a light bulb, or a supernova, spreads outward in 3D flat space, the total energy/sec transmitted passes through surfaces of larger and larger spheres. The area of those surfaces grows with the radial distance l_0 from the source by $4\pi l_0^2$. And so, luminosity decreases with the inverse square of the distance from the source.

Astronomers know the energy/sec emission rate of certain types of intergalactic objects, such as Type 1a supernovae. And so, by measuring the luminosity of such an object, one can, in principle, deduce the distance to the object.

The prior two paragraphs describe the situation for a flat 3D, static space. For an expanding universe like ours, or for a spatially curved universe, the situation changes. The following sub-sections can help in getting a handle on those changes.

10.3.2 The Three Basic Factors

As noted above, if space were static and flat, we would find light intensity (luminosity) diluted by the inverse square of the distance l_0 (again, the subscript 0 implies present time t_0) between us and the source. We assume energy from the source is radiated equally in all directions, such that the wave front forms greater and greater surfaces of spheres as time progresses. Note we can have a spherical shell of light in a 3D universe that is flat. Discern between the light wave front shape and the shape of space.

But, there are other factors at play, when the universe is expanding. The factors we need to consider are

1. Signal diminishing by increase in area over which energy/time is spread. In flat 3D space, this is inversely proportional to l_0^2, as discussed above. We hold off on discussing curved space.
2. Wavelength stretching, which reduces energy of each photon and thus reduces energy/time per unit area (luminosity).
3. Fewer photons per second passing by us, since the distance between photons increases as space expands.

10.3.3 For Flat Expanding 3D Space

1. For flat expanding space, we know l_0 as a function of the various ρ_I from (37) and the 1st Friedman equation, and luminosity (time rate of energy per unit area) is proportional to l_0^2.

   $$\frac{1}{l_0^2} = \text{ reduction in energy/sec per unit area from change in area due to expansion} \qquad (38)$$

2. The wavelength stretching factor is, employing (32),

   $$\frac{\lambda_0}{\lambda(t)} = \frac{a_0}{a(t)} = 1 + z\,. \qquad (39)$$

   The energy/sec per unit area is reduced by the inverse of (39),

   $$\frac{1}{1+z} = \text{ reduction in energy/time per unit area from } \lambda \text{ stretching via expansion} \qquad (40)$$

3. The number of photons passing us per unit time is reduced by the ratio of expansion lengths.

   $$\frac{1}{1+z} = \text{ reduction in energy/time per unit area from fewer photons/sec due to expansion} \qquad (41)$$

So, the total reduction in luminosity from all three sources is

$$\frac{1}{l_0^2\,(1+z)^2} = \text{ reduction in energy/sec per unit area from all three effects (flat space)} \qquad (42)$$

Thus, the denominator of (42) is the effective luminosity distance squared. It is not the actual physical distance l_0 to the source squared, but what seems to us like the square of the distance to the source if we were in a static universe.

$$d_L = (1+z)\,l_0 \qquad = \text{luminosity distance for flat, expanding, 3D universe}. \qquad (43)$$

10.3.4 For Curved Expanding 3D Space

From the RHS diagram of Fig. 3, which shows a positively curved 3D universe (with two dimensions suppressed), we can see that we find l_0 in the same was as 1) above. That is, both (34) and (37) remain valid. However, note that l_0 no longer represents the change in radius for the change in area (over which the energy/sec is distributed). The correct radial change is from l_{+1} to l_{0+1}, where the +1 subscript represents $k = +1$, positive curvature.

Thus, we have, where r_0 is the radius now of the positively curved universe,

$$l_{0+1} = r_0 \,sin \frac{l_0}{r_0}\,. \qquad (44)$$

Note that for a flat universe $r_0 = \infty$, and in the limit of $\rightarrow \infty$, (44) becomes $l_{0+1} = l_0$.

The negative curvature equivalent is harder to see graphically, so, we only state the result.

$$l_{0-1} = r_0 \,sinh \frac{l_0}{r_0}\,. \qquad (45)$$

The other two factors 2) and 3) above remain the same for curved spaces. Thus, the luminosity distance, in general, is

$$d_L = (1+z)\,l_0 \;\text{ flat} \qquad d_L = (1+z)\,l_{0+1} \;\text{ pos curvature} \qquad d_L = (1+z)\,l_{0-1} \;\text{ neg curvature}. \qquad (46)$$

10.4 The Plots

Hubble plots show z vs d_L. From (46), we get different plots for different curvatures and different mass-energy densities ρ_i in l_0. When astronomers plot values for z vs d_L they observe, they can discern the curvature of the universe and the amount of each ρ_I from the curve they get. See Fig. 4 for various mass-energy density amounts and curvatures, along with the data favoring the ΛCDM (Lambda cold dark matter, or concordance) model of 68% dark energy density (equivalently, a cosmological constant Λ contribution to density) and 32% matter (dark plus baryonic). Electromagnetic and neutrino densities are negligible.

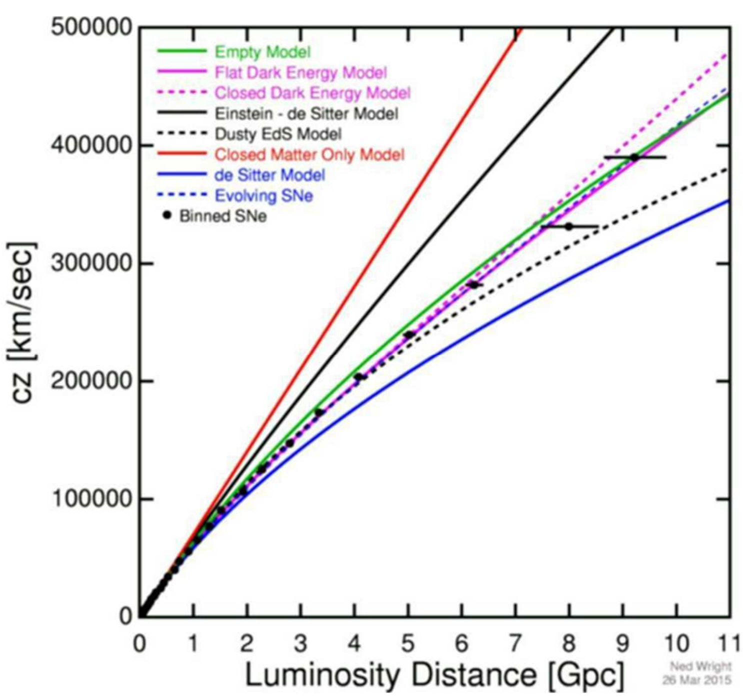


Figure 4. Hubble Chart Curves for Various Models of the Universe

H.W. Problem #7. Show using only the 1st Friedmann equation for a flat universe like ours, that for only mass density, and no vacuum mass-energy density, that the universe decelerates over time, approaching zero expansion velocity as $t \to \infty$. Note that it will never reverse direction and collapse, but only get smaller and smaller outward velocity with time.

Charts such as that of Fig. 4 for supernovae (of known intrinsic brightness) help determine the dark energy equation of state, which is often taken as the vacuum equation of state $p_{vac} = w\rho_{vac}$. The result is $w_{\text{vac}} = -1$, constant within a margin of error of a few percent over the history of universe.

10.5 The Hubble Tension

Readers have almost certainly heard of the Hubble tension, in which different measurements of the Hubble constant (at present time in the universe) give numbers that, given the experimental errors involved, are, as of 2025, too disparate. The method described herein, detecting luminosity and constructing Hubble plots such as those in Fig. 4, indicate that $H \cong 74$ km/sec/Mparsec. The cosmic microwave background radiation (CMBR) detection method, on the other hand, indicates $H \cong 67$ km/sec/Mparsec. The margins of error are well below the spread of approximately 7 km/sec/Mparsec.

No one knows, as of 2025, if this difference is due to a misunderstanding by us of the cosmological nature of the universe (implying a need for a new and better theory) or unknown experimental error, though the experimental and theoretical efforts trying to resolve the issue have been enormous.

11 A Big Crunch

One might ask how we could have a big crunch, a universe that slows down, stops expanding, starts contracting, and eventually ends up heading toward a singularity. That is, how, from our equations, do we get anything below the dashed line in the Fig. 1. This is the same as asking how we get the $\Omega_m = 6$ curve of Fig. 4.

We won't derive the dashed curve in Fig. 1, but simply look at one simplified case falling below the line, that of zero vacuum energy (zero cosmological constant) but mass density greater than critical.

$$\begin{gathered}\Omega_v = 0 \qquad \Omega_m > 1 \qquad \Omega = \Omega_v + \Omega_m > 1 \\ \left(\rho_{vac} = 0 \;\; \rho_{mass} > \rho_c \qquad \rho = \rho_{vac} + \rho_{mass} > \rho_c\right)\end{gathered} \tag{47}$$

Since total density is greater than critical, we must have positive curvature, i.e., $k = +1$. In the first Friedmann equation (A) on page one, we have, where K_1 is a constant and mass density is inversely proportional to a^3,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G\rho_{mass}}{3} - k\left(\frac{a_0}{a}\right)^2 = \frac{K_1}{a^3} - \left(\frac{a_0}{a}\right)^2 \quad \rightarrow \quad \dot{a}^2 = \frac{K_1}{a} - a_0^2 \tag{48}$$

$$\dot{a} = \pm\sqrt{\frac{K_1}{a} - a_0^2}\;. \tag{49}$$

Velocity is positive initially, but becomes zero when a is large enough to satisfy

$$\frac{K_1}{a} = a_0^2 \qquad\qquad a_{max} = \frac{K_1}{a_0^2}\,. \tag{50}$$

From the 2nd Friedmann equation, acceleration is always negative (= deceleration), so from this point on, velocity has to be directed back inward. That is, the negative sign in front of the square root symbol in (49) holds.

Note that if we had less than critical density, $k = -1$. For all such cases, instead of (49), we get

$$\dot{a} = \pm\sqrt{\frac{K_1}{a} + a_0^2}\;, \tag{51}$$

which is always a positive number, no matter how large a grows. Velocity of expansion will tend towards zero, but never reach it, and never become negative. Note from Fig. 1, for zero vacuum density (the horizontal axis) and mass density less than critical, we have deceleration forever, but expansion never stops.

This was the typical picture one saw in relativity texts prior to the discovery of dark energy in 1998. Everyone thought mass density was the only actor in the play.

H.W. Problem #8. Where on the plot of Fig. 1 will our universe be billions of years into the future? Qualitative answer only. No calculations. In the past, were we ever in a deceleration phase? If your answer is yes (it should be), show where that region would be on the plot.

H.W. Problem #9. Note which curve in Fig. 2 is our universe. Explain why its slope early on indicates deceleration and why, well into the future, the slope indicates acceleration. Hint: Consider what component of ρ is dominating at each time and what its effect on the time derivation of a is, as shown in Sect. 8.

12 Homework Solutions

H.W. Problem #1: Derive handwritten (3) above using the energy balance relation $d\left(\rho V\right)=-pdV$ where $V \alpha\, a^3$.

$$d\left(\rho V\right)=-pdV \qquad\qquad V\propto Ka^3 \tag{52}$$

$$\frac{d\left(\rho Ka^3\right)}{dt}=-p\frac{d\left(Ka^3\right)}{dt} \quad\rightarrow\quad \dot{\rho}a^3+3\rho a^2\dot{a}=-3a^2\dot{a} \quad\rightarrow\quad \dot{\rho}a+3\rho\dot{a}=-3\dot{a} \tag{53}$$

H.W. Problem #2: Show that the energy balance equation holds for 1) for mass, where $p = 0$, $\rho\ \alpha\ 1/a^3$; 2) for radiation, where $p = \rho/3$, $\rho\ \alpha\ 1/a^3$ 3) for the vacuum, assuming $p = -\rho$, ρ = constant, and 4) for any substance where the equation of state is $p = w\rho$ (w a constant), $\rho\ \alpha\ a^{3(1+w)}$.

$$\dot{\rho}=-3\left(\rho+p\right)\dot{a}/a \tag{54}$$

1) In (54), $\dot{\rho}=-3\rho\dot{a}/a \rightarrow \dfrac{d}{dt}\left(\dfrac{K}{a^3}\right)a=-3\dfrac{K}{a^3}\dot{a} \quad\rightarrow\quad -3a^{-4}\dot{a}a=-3\dfrac{\dot{a}}{a^3} \quad\rightarrow\quad -\dot{a}=-\dot{a}$

2) You can do this.

3) You can do this.

4) In (54),
$$\dot{\rho}=-3\left(\rho+w\rho\right)\dot{a}/a \rightarrow \frac{d}{dt}\left(a^{-3(1+w)}\right)a=-3\left(a^{-3(1+w)}+w\left(a^{-3(1+w)}\right)\right)\dot{a}$$
$$=-3(1+w)a^{-3(1+w)-1}\dot{a}a=-3(1+w)a^{-3(1+w)}\dot{a} \quad\rightarrow\quad \dot{a}=\dot{a}$$

H.W. Problem #3. Show that, for matter, $a\ \alpha\ t^{2/3}$

This problem is not difficult, so you can do it on your own.

H.W. Problem #4. Show that for a radiation dominated flat universe, even though its size increases toward infinity as $t\rightarrow\infty$, the speed of expansion goes to zero. Then, show the same thing happens, qualitatively, though at different rates, for a matter dominated flat universe. Things are different for a curved universe where $k \neq 0$, since (19) no longer holds. It has an extra term with a factor of k in it.

From (20), for a radiation dominated universe,

$$\dot{a}=C\sqrt{\frac{8\pi G}{3}}\frac{1}{a}, \qquad\qquad \text{repeat of (20)}$$

as $a\rightarrow\ \infty$, the time derivative of $a\rightarrow 0$.

H.W. Problem #5. Show how the dynamics and geometry of the early universe (post inflation) must have been dominated by radiation mass-density. Then show how, subsequently, the universe, for quite some time, was dominated by mass. And how, in the far future, it will be dominated by the cosmological constant (source = dark energy), or in other words, by vacuum mass-energy density (at least it can be modeled that way).

From (19), where ρ can be from matter, radiation, or a cosmological constant,

$$\dot{a} = \sqrt{\frac{8\pi G\rho}{3}}a \,. \tag{55}$$

Due to any one of radiation, matter, or vacuum (cosmological constant), we have

$$\text{Radiation: } \rho_\gamma \propto \frac{1}{a^4} \qquad \text{Mass: } \rho_m \propto \frac{1}{a^3} \qquad \text{Cosmological constant: } \rho_{cc} = \text{ constant }. \tag{56}$$

At the beginning of the universe, assuming relatively comparable total amounts of mass-energy from radiation and mass, along with a small density for the c.c., with a very small, radiation mass-energy density would be greater than the other sources, and dominate in (55). As a grew larger, radiation density would be reduced much faster than mass density until at some point, it mass density became larger than radiation density. The difference would increase over time until radiation became negligible compared to mass. At this point the c.c. density was still small relative to mass density. However, over time, as a grew, mass desnity got smaller and smaller, where eventually it was less than the c.c. density. As time went on the influence of the c.c. density would become greater and greater, compared with mass density. And that is where we are today. The c.c. density dominates mass density, with radiation density virtually negligible.

H.W. Problem #6. Start with the 2nd Friedmann equation (18), but assume the cosmological constant part with Λ is included in the mass-energy density ρ and pressure p, so there is no Λ term. Then show, from that equation, that we get

Universe deceleration if ρ + 3p > 0. Universe acceleration if ρ + 3p < 0.

Then, show, if we have positive energy density for the vacuum (i.e., for dark energy, positive Λ) with no mass, then we have negative pressure, and the universe will accelerate.

Then show, for the present day, where mass density is 32% and dark energy density is 68%, that we have acceleration.

Then show for what values of p, relative to ρ, do we have acceleration or deceleration.

Then, note how this shows up in Fig. 1 below (and also on pg. 1 near the bottom).

In (18), re-arranged, $\ddot{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right)a$, a is always positive. So ρ + 3p > 0, means deceleration. $\ddot{a}$ is negative, i.e., the universe decelerates. ρ + 3p < 0, means acceleration, since $\ddot{a}$ is positive.

For dark energy p = $-\rho$, where ρ is positive. So, $\rho + 3p = \rho - 3\rho = -2\rho < 0$, and $\ddot{a}$ is positive, i.e., the universe accelerates.

For the present day, mass pressure is effectively = 0. Total density $\rho = \rho_{mass} + \rho_{vac} = \rho_c$, since the universe is 3D flat, so mass density must be critical mass density. Then, for 32% matter and 68% dark energy, $\rho = .32\,\rho_c + .68\,\rho_c = \rho_c$. $p_{vac} = -\rho_{vac} = -.68\,\rho_c$. So, $\rho + 3p = \rho_{mass} + \rho_{vac} + 3p_{vac} = \rho_c + 3(-.68\,\rho_c) = -1.04 < 0$. So, acceleration is positive now.

For $p < -\rho/3$, ρ + 3p < 0, and we have acceleration.

H.W. Problem #7. Show using only the 1st Friedmann equation for a flat universe like ours, that for only mass density, and no vacuum mass-energy density, that the universe decelerates over time, approaching zero expansion velocity as $t \to \infty$. Note that it will never reverse direction and collapse, but only get smaller and smaller outward velocity with time.

From (55) above (the Friedmann equation with k = 0 and Λ = 0), with the mass density of (56),

$$\dot{a} \propto \sqrt{\frac{8\pi G}{3a^3}}a = \sqrt{\frac{8\pi G}{3a}} \,. \tag{57}$$

As $a \to \infty$, the rate of expansion, the time derivative of $a \to 0$. The universe decelerates. But the velocity never actually gets to zero, and it never becomes negative (towards collapse of the universe). It just keeps expanding, but slower and slower.

H.W. Problem #8. Where on the plot of Fig. 1 will our universe be billions of years into the future? Qualitative answer only. No calculations. In the past, were we ever in a deceleration phase? If your answer is yes (it should be), show where that region would be on the plot.

A flat universe, as it expands, stays flat. It follows the line of $\Omega = 1$ (critical density means flat). As the universe expands, more and more of the total mass-energy density will be from the vacuum (the c.c.), since that stays constant, while mass density decreases with a^3. So, we will approach $\Omega_V = 1$. That is, we will move up the straight line where the dot "Us now" is on towards the "1" value on the vertical axis.

Yes, we were once in a deceleration phase, when mass energy density dominated over vacuum (c.c.) mass-energy. Thus, we were on the same line stretching from1 on the horizontal axis to 1 on the vertical axis, but below the line stretching from the origin up and to the right.

Note that (as lengthy calculations using observational data show) the two line intersected about 6 billion years after the Big Bang.

H.W. Problem #9. Note which curve in Fig. 2 is our universe. Explain why its slope early on indicates deceleration and why, well into the future, the slope indicates acceleration. Hint: Consider what component of ρ is dominating at each time and what its effect on the time derivation of a is, as shown in Sect. 8.

The top curve is our universe, which presently has about 30% (actually 32% to better accuracy) matter (normal plus dark) density and about 70% (68% more accurately) vacuum mass-energy density, with, for all practical purposes, zero radiation mass-energy density.

Early on, matter dominated, so the universe was decelerating. The average distance between galaxies was increasing, but at a slower and slower rate. The slope of the curve, the time derivative of the average distance between galaxies was becoming less. The slope of the curve was gradually becoming less and less. But, when dark energy (the vacuum, the c.c.) became more dominant, the rate of change of distance increased over time. The slope got bigger and bigger a time passed.

[1] T.F. Jordan, Cosmology calculations almost without general relativity, *Am J Phys* **73** (7) July 2005, 653-662. Note he uses R where we use a (the latter is more common in the literature). https://pubs.aip.org/aapt/ajp/article-abstract/73/7/653/1056176/Cosmology-calculations-almost-without-general?redirectedFrom=fulltext